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Boundary generating curves of the c -numerical range

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Abstract

Let A be an $n \times n$ matrix and $c = (c_1, c_2, \dots, c_n)$ be a real n -tuple. The c -numerical range of A is the set $W_c(A) = \{ \sum_{j=1}^n c_j x_j^* A x_j : \{x_1, x_2, \dots, x_n\} \text{ is an orthonormal basis of } \mathbb{C}^n \}$. We obtain parametric representations of the boundary generating curve of the c -numerical range of a matrix. Applying this result, we generalize the result of Anderson to the c -numerical range. Furthermore, we give a description of the boundary generating curve of the c -numerical range of certain types of nilpotent Toeplitz matrices. A sufficient condition for the boundary generating curve to be rational is obtained. Finally we explicitly compute the boundary generating curves of the numerical ranges for several concrete matrices and classify the rationality of the curves. © 1999 Elsevier Science Inc. All rights reserved.

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1. Introduction

For an $n \times n$ complex matrix A , the numerical range of A is the set of complex numbers

$$W(A) = \{x^*Ax : x \in \mathbb{C}^n, |x| = 1\}.$$

The numerical range of a matrix is a classical object in matrix analysis. It is well known that $W(A)$ is a convex set. We list some basic facts concerning the shape of the numerical range:

- (i) The numerical range of a normal matrix is the convex hull of its eigenvalues.
- (ii) The numerical range of a 2×2 matrix is an elliptical disc.
- (iii) The numerical range of an upper triangular nilpotent matrix associated with a tree graph is a circular disc.

The above three properties show that under certain conditions the boundary of the numerical range of a matrix is the following arc γ or the finite union of such arcs

$$\gamma = \{\phi(\theta) + i\psi(\theta) : \alpha \leq \theta \leq \beta\},$$

where $\phi(\theta)$ and $\psi(\theta)$ are real trigonometric polynomials. A sequence of related papers [6,8,16] discussed conditions for $W(A)$ to be a circular disc or an elliptical disc. However, it is still an open problem to characterize which compact convex subset of \mathbb{R}^2 , identified with \mathbb{C} , being the numerical range of a matrix. In this paper, we study parametric representations and rationality of the boundary generating curve of the numerical range.

A generalization of the numerical range, introduced by Westwick [19] is the c -numerical range of $A \in M_n$ with $c = (c_1, c_2, \dots, c_n) \in \mathbb{R}^n$, defined by

$$W_c(A) = \left\{ \sum_{j=1}^n c_j x_j^* A x_j : \{x_1, x_2, \dots, x_n\} \text{ is an orthonormal basis of } \mathbb{C}^n \right\}.$$

Westwick showed that the set $W_c(A)$ is convex. Since $W_c(A)$ remains unchanged when we reorder the coordinates of c , we assume, in this paper, that the coordinates of c are decreasing. In Section 2, we obtain parametric representations of the boundary generating curve of the c -numerical range of A . Applying this result, we generalize a theorem of Anderson to the c -numerical range. Indeed we prove, in Theorem 2.2, that if $c \in \mathbb{R}^n$ has distinct entries $c_1 > c_2 > \dots > c_k$ with multiplicities m_1, m_2, \dots, m_k , respectively and if $W_c(A)$ is contained in a closed unit disc D and contains at least $n!/(m_1!, \dots, m_k!) + 1$ points of the boundary of D , then $W_c(A) = D$. Our method used in this section is based on Kippenhahn's classical idea [12] and Gårding's theory of hyperbolic polynomials [1]. In Section 3, we treat a certain type of nilpotent Toeplitz matrix B given by (3.1) or (3.3). The Hermitian part of the matrix $(\exp(i\theta))B$

becomes a generalized circulant matrix. We prove that the c -numerical range of B is represented by the convex hull of the classical numerical ranges of some matrices of the same type. We also obtain a necessary and sufficient condition so that the graph corresponding to B is connected. In this case, we show that the numerical range of B is the convex hull of a rational curve. Finally, in Section 4, we consider several concrete matrices, parametrize the boundary generating curves of the numerical ranges, and discuss rationality of the generating curves.

2. Parametrization of the range

Let A be an $n \times n$ matrix. The homogeneous polynomial associated with A is defined by

$$h(t, x, y) = \det(tI_n + x(A + A^*)/2 - iy(A - A^*)/2). \quad (2.1)$$

Let Γ_h be the algebraic curve of $h(t, x, y)$, i.e.,

$$\Gamma_h = \{[(t, x, y)] \in \mathbb{CP}^2 : h(t, x, y) = 0\}, \quad (2.2)$$

where $[(t, x, y)]$ is the equivalence class containing $(t, x, y) \in \mathbb{C}^3 - (0, 0, 0)$ under the relation $(t_1, x_1, y_1) \sim (t_2, x_2, y_2)$ if $(t_2, x_2, y_2) = k(t_1, x_1, y_1)$ for some non-zero complex number k . The dual curve of Γ_h is defined by

$$\Gamma_h^\wedge = \{[(T, X, Y)] \in \mathbb{CP}^2 : Tt + Xx + Yy = 0 \text{ is a tangent line of } \Gamma_h\}. \quad (2.3)$$

Kippenhahn [12] showed that $W(A)$ is the convex hull of Γ_h^\wedge in the real affine plane. There have been several papers on the numerical range along Kippenhahn's direction (cf. [10, 11, 17, 18]).

For each x, y , we denote the n real roots of $h(t, x, y) = 0$ by

$$\lambda_1(x, y) \geq \lambda_2(x, y) \geq \cdots \geq \lambda_n(x, y).$$

Let $c \in \mathbb{R}^n$ have distinct entries $c_1 > c_2 > \cdots > c_k$ with multiplicities m_1, m_2, \dots, m_k , respectively. Denote by $P(m_1, \dots, m_k)$ the collection of (M_1, \dots, M_k) such that M_1, \dots, M_k form a partition of $\{1, 2, \dots, n\}$ so that M_j has m_j elements. Define the homogeneous polynomial

$$H(t, x, y) = \prod_{(M_1, \dots, M_k) \in P(m_1, \dots, m_k)} \left(t - \sum_{j=1}^k c_j \sum_{s \in M_j} \lambda_s(x, y) \right). \quad (2.4)$$

Then $H(t, x, y)$ is a polynomial in t of degree

$$\binom{n}{m_1 \ m_2 \ \cdots \ m_k} = \frac{n!}{m_1! \ m_2! \ \cdots \ m_k!}$$

and the coefficients of t are symmetric functions of $\lambda_s(x, y)$. We define Γ_H and Γ_H^\wedge similar to that of (2.2) and (2.3). In this section, we generalize Kippenhahn's result to the c -numerical range in Theorem 2.1. Applying Theorem 2.1, we extend Anderson's result in Theorem 2.2.

Theorem 2.1. *Let A be an $n \times n$ complex matrix and $c = (c_1, \dots, c_n) \in \mathbb{R}^n$. Suppose that H is the homogeneous polynomial given by Eq. (2.4). Then*

$$W_c(A) = \text{conv}(\{X + iY : (X, Y) \in \mathbb{R}^2, t + Xx + Yy = 0 \\ \text{is a tangent line of } \Gamma_H\}).$$

Proof. Clearly, the polynomial $H(t, x, y)$ is hyperbolic with respect to the vector $(1, 0, 0)$. Then by a result [12, p. 201], the order of every prime point of the curve Γ_H with real center is 1 and hence every real singular point of Γ_H is an ordinary multiple point or a tacnode. Denote by Ω the connected component containing $(1, 0, 0)$ of the open set

$$\{(t, x, y) \in \mathbb{R}^3 \setminus \{(0, 0, 0)\} : H(t, x, y) \neq 0\}.$$

Then Ω is a convex cone with vertex $(0, 0, 0)$. If $L : at + bx + cy = 0$ is a real tangent line of Γ_H , then by [1, Corollary 3.2.3] the line L does not pass through any point of Ω .

The dual cone Ω^\wedge of Ω is denoted by

$$\Omega^\wedge = \{(T, X, Y) \in \mathbb{R}^3 : Tt + Xx + Yy \geq 0 \text{ for every } (t, x, y) \in \Omega\}.$$

If Ω^\wedge has no interior point, then the polynomial $H(t, x, y)$ is the product of linear factors $t + p_jx + q_jy$. In such a case, the range $W_c(A)$ is the convex hull of the points $p_j + \sqrt{-1}q_j$ on the Gaussian plane and the assertion follows.

If Ω^\wedge has an interior point, then there exists a plane Π in \mathbb{R}^3 with $(0, 0, 0) \notin \Pi$ such that

$$\Omega = \{a(t, x, y) : a > 0, (t, x, y) \in \Pi \cap \Omega\}$$

and the intersection $\Pi \cap \Omega$ is bounded. By the separation theorem of compact convex sets, the set

$$\{(X, Y) \in \mathbb{R}^2 : (1, X, Y) \in \Omega^\wedge\}$$

is the convex hull of the set

$$\{(X, Y) \in \mathbb{R}^2 : t + Xx + Yy = 0 \text{ is a tangent line of } \Gamma_H \\ \text{at some point of } \partial\Omega\}.$$

Suppose that $(1, 0, 0)$ is an interior point of Ω , or equivalently, for every $(x, y) \in \mathbb{R}^2 \setminus \{(0, 0)\}$, there exists $t > 0$ such that $H(t, x, y) = 0$. Then $(1, x, y) \in \partial\Omega$ is equivalent to the condition

$$1 = \max\{t \in \mathbb{R} : H(t, x, y) = 0\}.$$

Now by a result [15],

$$\sum_{j=1} c_j \lambda_j(x, y)$$

is the maximum of real numbers t satisfying $H(t, x, y) = 0$. Hence the convex curve

$$\partial\Omega_1 = \{(x, y) \in \mathbb{R}^2 : (1, x, y) \in \partial\Omega\}$$

is parametrized by

$$\left\{ \left(- \left(\sum_{j=1} c_j \lambda_j(\cos \theta, \sin \theta) \right)^{-1} \cos \theta, \left(\sum_{j=1} c_j \lambda_j(\cos \theta, \sin \theta) \right)^{-1} \sin \theta \right) : 0 \leq \theta \leq 2\pi \right\}. \quad (2.5)$$

Applying a result of Li–Sung–Ting [15] that

$$x \cos \theta - y \sin \theta - \sum_{j=1} c_j \lambda_j(\cos \theta, \sin \theta) = 0$$

is a supporting line of $W_c(A)$, we obtain that $W_c(A)$ is the convex set bounded by the curve

$$\{(X + iY) : (X, Y) \in \mathbb{R}^2, Xx + Yy + 1 = 0 \text{ is a tangent line of } \partial\Omega_1\}$$

and this completes the proof. \square

From a geometric viewpoint, Joel Anderson in 1970 showed that if $W(A)$ is contained in a closed unit disc and contains at least $n + 1$ points of the boundary of the disc, then $W(A)$ is the unit disc.

As a consequence of Theorem 2.1, we generalize Anderson's result to the c -numerical range.

Theorem 2.2. *Let A be an $n \times n$ complex matrix, and let $c \in \mathbb{R}^n$ have distinct entries $c_1 > c_2 > \cdots > c_k$ with multiplicities m_1, m_2, \dots, m_k , respectively. If $W_c(A)$ is contained in a closed unit disc D and contains at least*

$$\binom{n}{m_1 \ m_2 \ \cdots \ m_k} + 1$$

points of the boundary of D , then $W_c(A) = D$.

Proof. By a result [1], the polynomial $H(t, x, y)$ defined by (2.4) is hyperbolic with respect to the point $(1, 0, 0)$ and for every real pair x, y , $H(t, x, y) = 0$ has

$$q \equiv \binom{n}{m_1 \ m_2 \ \cdots \ m_k}$$

real roots. Suppose that $W_c(A)$ is contained in $D = \{z \in \mathbb{C} : |z| \leq 1\}$ and contains the points $\exp(i\theta_j)$, $0 \leq \theta_1 < \theta_2 < \cdots < \theta_{q+1} < 2\pi$. Note that the curve

$$\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$$

is self-dual. By the relation that $W_c(A) \subseteq D$ and the duality of convex sets, it follows that the two curves

$$\Gamma_1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$$

and

$$\Gamma_2 = \{(x, y) \in \mathbb{R}^2 : H(1, x, y) = 0\}$$

satisfy the relation

$$\{(x, y) \in \mathbb{R}^2 : H(1, x, y) = 0\} \subseteq \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \geq 1\}.$$

Moreover, for every $j = 1, 2, \dots, q+1$, the point $P_j = (-\cos \theta_j, -\sin \theta_j)$ is a common of Γ_1 and Γ_2 and the two curves have the common tangent $(\cos \theta_j)x + (\sin \theta_j)y + 1 = 0$ at the point P_j . The number of intersections of Γ_1 and Γ_2 at P_j is not less than 2. By the Bezout theorem, Γ_1 and Γ_2 have an irreducible common component. But since Γ_1 is irreducible, it follows that $\Gamma_1 \subseteq \Gamma_2$. Then by Theorem 2.1 and the duality of convex sets, we have $W_c(A) = D$. \square

Remark. If $c = (1, 0, \dots, 0)$, the polynomial $H(t, x, y)$ (2.4) reduces to the polynomial (2.1) with degree n . The quantity

$$\binom{n}{m_1 \ m_2 \ \cdots \ m_k} + 1$$

becomes $n+1$, Theorem 2.2 is a generalization of the result of Anderson.

Whether the number

$$\binom{n}{m_1 \ m_2 \ \cdots \ m_k} + 1$$

in Theorem 2.2 is sharp is unknown. However it is the best possible for $n = 2, 3$. Clearly, the number $3 = (2!/1!1!) + 1$ is the best possible for $n = 2$. When $n = 3$, we consider the matrix $A = (1/\sqrt{3}) \operatorname{diag}(1, \omega, \bar{\omega})$,

$\omega = \exp(2\pi i/3)$, and $c = (1, 0, -1)$. By a result [15], $W_c(A)$ is the hexagon formed by the 6 vertices

$$(1 - \omega)/\sqrt{3}, (\omega - 1)/\sqrt{3}, (1 - \bar{\omega})/\sqrt{3}, (\bar{\omega} - 1)/\sqrt{3}, (\omega - \bar{\omega})/\sqrt{3}$$

and $(\bar{\omega} - \omega)/\sqrt{3}$ on the unit circle. This shows that $7 = (3!/1!1!1!) + 1$ is the best possible for $n = 3$.

Theorem 2.1 yields another proof of the following known result.

Theorem 2.3 (cf. [4]). *Let A be an $n \times n$ matrix and $c = (c_1, c_2, \dots, c_n) \in \mathbb{R}^n$. Suppose that*

$$\lambda_1(\theta) \geq \lambda_2(\theta) \geq \dots \geq \lambda_n(\theta)$$

are roots of the characteristic polynomial

$$\det(\lambda I_n - \cos \theta(A + A^*)/2 + \sin \theta(A - A^*)/(2i)) = 0,$$

and $\lambda(\theta)$ is the continuous function

$$\lambda(\theta) = \sum_{j=1}^n c_j \lambda_j(\theta).$$

Then the c -numerical range $W_c(A)$ is the convex hull of the curve $\{X(\theta) + iY(\theta) : 0 \leq \theta \leq 2\pi\}$ where $X(\theta), Y(\theta)$ are given by

$$X(\theta) = \cos \theta \lambda(\theta) - \sin \theta \lambda'(\theta),$$

$$Y(\theta) = -\sin \theta \lambda(\theta) - \cos \theta \lambda'(\theta).$$

Proof. From (2.5) the curve $\partial\Omega_1$ is parametrized by

$$x = x(\theta) = -\frac{\cos \theta}{\lambda(\theta)}, \quad y = y(\theta) = \frac{\sin \theta}{\lambda(\theta)}.$$

Then the dual curve of $\partial\Omega_1$ is given by

$$X(\theta) = \frac{-y'(\theta)}{x(\theta)y'(\theta) - y(\theta)x'(\theta)},$$

$$Y(\theta) = \frac{x'(\theta)}{x(\theta)y'(\theta) - y(\theta)x'(\theta)}. \quad \square$$

The curve $\{X(\theta) + iY(\theta) : 0 \leq \theta \leq 2\pi\}$ in Theorem 2.3 is called the *boundary generating curve* of $W_c(A)$.

3. Nilpotent Toeplitz matrices

Consider an $n \times n$ complex matrix of the form

$$\begin{pmatrix} a_1 & a_2 & a_3 & \dots & a_n \\ \alpha a_n & a_1 & a_2 & \dots & a_{n-1} \\ \alpha a_{n-1} & \alpha a_n & a_1 & \dots & a_{n-2} \\ \dots & \dots & \dots & \dots & \dots \\ \alpha a_2 & \alpha a_3 & \alpha a_4 & \dots & a_1 \end{pmatrix},$$

where $\alpha \neq 0$ and a_1, \dots, a_n are complex numbers. This type of matrix is called the *generalized circulant matrix*. It is shown in [9] that a normal Toeplitz matrix is either a generalized circulant matrix or is obtained from a Hermitian Toeplitz matrix by rotation and translation. When $\alpha = 1$, the generalized circulant matrix reduces to the standard circulant matrix. Circulant matrices play an important role in solving Toeplitz systems (cf. [2,3]).

Suppose that $n = 2m$ is an even number ≥ 2 , and $\beta_1, \dots, \beta_{m-1}$ are arbitrary complex numbers and $\beta_m \in \mathbb{R}$. We define the following $n \times n$ nilpotent Toeplitz matrix

$$B = \begin{pmatrix} 0 & \beta_1 & \beta_2 & \dots & \beta_{m-1} & \beta_m & \overline{\beta_{m-1}} & \dots & \overline{\beta_2} & \overline{\beta_1} \\ 0 & 0 & \beta_1 & \dots & \beta_{m-2} & \beta_{m-1} & \beta_m & \dots & \overline{\beta_3} & \overline{\beta_2} \\ 0 & 0 & 0 & \dots & \beta_{m-3} & \beta_{m-2} & \beta_{m-1} & \dots & \overline{\beta_4} & \overline{\beta_3} \\ 0 & 0 & 0 & 0 & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & \beta_1 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \end{pmatrix}. \quad (3.1)$$

Denote this matrix B by $A(\beta_1, \dots, \beta_{m-1}, \beta_m)$. Consider the Hermitian matrix

$$H(\theta) = (\exp(i\theta)B)_h = (e^{i\theta}B + e^{-i\theta}B^*)/2, \quad (3.2)$$

$\theta \in \mathbb{R}$. Then $2H(\theta)$ becomes

$$\begin{pmatrix} 0 & \exp(i\theta)\beta_1 & \exp(i\theta)\beta_2 & \dots & \exp(i\theta)\overline{\beta_2} & \exp(i\theta)\overline{\beta_1} \\ \exp(-i\theta)\overline{\beta_1} & 0 & \exp(i\theta)\beta_1 & \dots & \exp(i\theta)\overline{\beta_3} & \exp(i\theta)\overline{\beta_2} \\ \exp(-i\theta)\overline{\beta_2} & \exp(-i\theta)\overline{\beta_1} & 0 & \dots & \exp(i\theta)\overline{\beta_4} & \exp(i\theta)\overline{\beta_3} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \exp(-i\theta)\beta_1 & \exp(-i\theta)\beta_2 & \exp(-i\theta)\beta_3 & \dots & \exp(-i\theta)\overline{\beta_1} & 0 \end{pmatrix}$$

and $2 \exp(-i\theta)H(\theta)$ is the generalized circulant matrix

$$\begin{pmatrix} 0 & \beta_1 & \beta_2 & \cdots & \overline{\beta_2} & \overline{\beta_1} \\ \alpha\overline{\beta_1} & 0 & \beta_1 & \cdots & \overline{\beta_3} & \overline{\beta_2} \\ \alpha\overline{\beta_2} & \alpha\overline{\beta_1} & 0 & \cdots & \overline{\beta_4} & \overline{\beta_3} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \alpha\beta_1 & \alpha\beta_2 & \alpha\beta_3 & \cdots & \alpha\overline{\beta_1} & 0 \end{pmatrix}$$

with $\alpha = \exp(-2i\theta)$. For $j = 1, 2, \dots, m$, we write $\beta_j = a_j + ib_j, a_j, b_j \in \mathbb{R}$. Then, by the formula of eigenvalues of a generalized circulant matrix in [13, p. 66], the eigenvalues $\lambda_k(\theta)$ of $H(\theta)$ are given by

$$\lambda_k(\theta) = (-1)^k(1/2)a_m + (-1)^k \Re \left(\sum_{j=1}^{m-1} \beta_{m-j} \exp \left(i \left(\frac{2j}{n} \theta - \frac{2jk\pi}{n} \right) \right) \right),$$

$k = 0, 1, 2, \dots, n-1$.

Suppose that $n = 2m - 1$ is an odd number ≥ 3 , and $\beta_1, \dots, \beta_{m-1}$ are arbitrary complex numbers. We define an $n \times n$ nilpotent Toeplitz matrix

$$B = A(\beta_1, \dots, \beta_{m-1}) \quad (3.3)$$

in a similar fashion of (3.1). Then the eigenvalues $\lambda_k(\theta)$ of the corresponding Hermitian matrix $H(\theta)$ are

$$\lambda_k(\theta) = (-1)^k \Re \left(\sum_{j=1}^{m-1} \beta_{m-j} \exp \left(i \left(\frac{2j-1}{n} \theta - \frac{(2j-1)k\pi}{n} \right) \right) \right)$$

for $k = 0, 1, 2, \dots, n-1$.

In summary, we obtain the following result.

Theorem 3.1. *Let B be a nilpotent Toeplitz matrix given by (3.1) or (3.3) and $H(\theta)$ be the Hermitian matrix defined by (3.2). Then the eigenvalues $\lambda_k(\theta)$ of $H(\theta)$ are given by the following:*

(I) For $n = 2m$,

$$\lambda_k(\theta) = (-1)^k(1/2)a_m + (-1)^k \Re \left(\sum_{j=1}^{m-1} \beta_{m-j} \exp \left(i \left(\frac{2j}{n} \theta - \frac{2jk\pi}{n} \right) \right) \right), \quad (3.4)$$

$k = 0, 1, 2, \dots, n-1$.

(II) For $n = 2m - 1$,

$$\lambda_k(\theta) = (-1)^k \Re \left(\sum_{j=1}^{m-1} \beta_{m-j} \exp \left(i \left(\frac{2j-1}{n} \theta - \frac{(2j-1)k\pi}{n} \right) \right) \right), \quad (3.5)$$

$k = 0, 1, 2, \dots, n-1$.

For every $k \in \mathbb{Z}$, we consider $\lambda_k(\theta)$ given by (3.4) or (3.5) according to n is even or odd. By direct computations, we find that $\lambda_k(\theta) = \lambda_\ell(\theta)$ holds for every θ if and only if

$$(k - \ell)j \equiv 0 \pmod{n\mathbb{Z}}$$

for every j with $\beta_j \neq 0$. Thus the sequence of functions $\{\lambda_k(\theta) : k \in \mathbb{Z}\}$ has a least period $n_0 = n/Q$, where Q is the generator of the ideal

$$j_1\mathbb{Z} + j_2\mathbb{Z} + \cdots + j_p\mathbb{Z} + n\mathbb{Z}$$

of \mathbb{Z} , and

$$\{j : 1 \leq j \leq [n/2], \beta_j \neq 0\} = \{j_1 < j_2 < \cdots < j_p\}.$$

Next we consider the graph corresponding to the matrix $B = A(\beta_1, \beta_2, \dots)$. We see easily that the graph is connected if and only if the ideal $Q\mathbb{Z}$ coincides with \mathbb{Z} . If $Q > 1$, then the graph is decomposed into Q copies of a connected circulant graph with n_0 vertices. In this case the matrix B is the direct sum of Q copies of an $n_0 \times n_0$ nilpotent Toeplitz matrix of the form (3.1) or (3.3). We prove a stronger assertion.

Theorem 3.2. *Let B be a nilpotent Toeplitz matrix given by (3.1) or (3.3) and $F(t, x, y) \in \mathbb{C}[t, x, y]$ be the real homogeneous polynomial of degree n defined by*

$$F(t, x, y) = \det(tI_n + x(B + B^*)/2 - iy(B - B^*)/2).$$

Then the polynomial F is irreducible in the polynomial ring $\mathbb{C}[t, x, y]$ if and only if the graph corresponding to the matrix B is connected. In this case, the complex projective curve Γ given by

$$\Gamma = \{[(t, x, y)] \in \mathbb{CP}^2 : F(t, x, y) = 0\}$$

is a rational curve.

Proof. If the graph of B is disconnected, or equivalently, if $Q > 1$, then B is the direct sum of Q copies of an $n_0 \times n_0$ nilpotent matrix B_1 . Define

$$F_1(t, x, y) = \det(tI_{n_0} + x(B_1 + B_1^*)/2 - iy(B_1 - B_1^*)/2).$$

Then we have

$$F(t, x, y) = F_1(t, x, y)^Q$$

and hence F is reducible. Conversely, we assume that $Q = 1$. Then

$$F(t, -\cos \theta, \sin \theta) = \prod_{k=0}^{n-1} (t - \lambda_k(\theta))$$

for every $\theta \in \mathbb{R}$. By (3.4) and (3.5), we have the relation

$$\lambda_k(\theta) = (-1)^k \lambda_0(\theta - k\pi) \quad (3.6)$$

and thus

$$\begin{aligned} & [(\lambda_k(\theta), -\cos \theta, \sin \theta)] \\ &= [((-1)^k \lambda_0(\theta - k\pi), -(-1)^k \cos(\theta - k\pi), (-1)^k \sin(\theta - k\pi))] \\ &= [(\lambda_0(\theta - k\pi), -\cos(\theta - k\pi), \sin(\theta - k\pi))]. \end{aligned}$$

Hence every point of Γ can be expressed as $[(\lambda_0(\theta), -\cos \theta, \sin \theta)]$ for some $\theta \in \mathbb{R}$. An affine form of Γ is given by

$$\{(x, y) \in \mathbb{R}^2 : F(1, x, y) = 0\}$$

which can be parametrized by

$$\left\{ \left(-\frac{\cos \theta}{\lambda_0(\theta)}, \frac{\sin \theta}{\lambda_0(\theta)} \right) : \theta \in \mathbb{R}, \lambda_0(\theta) \neq 0 \right\}. \quad (3.7)$$

Since $\lambda_0(\theta)$ is a trigonometric polynomial in the variable θ/n , the functions $-\cos \theta/\lambda_0(\theta)$ and $\sin \theta/\lambda_0(\theta)$ can be expressed as rational functions with real coefficients in the variable $\tan(\theta/(2n))$. Thus there exists an irreducible polynomial $F_0(t, x, y)$ and a natural number p such that the curve $F_0(t, x, y) = 0$ is rational and $F = F_0^p$. Since $\lambda_k(\theta)$ ($k = 0, 1, \dots, n-1$) of Theorem 3.1 are mutually distinct analytic functions, then by (3.6) there exists $\theta_0 \in \mathbb{R}$ such that

$$F_t(\lambda_0(\theta_0), -\cos \theta_0, \sin \theta_0) \neq 0.$$

This implies that $p = 1$ and thus F is irreducible and Γ is a rational curve. \square

Let $A = A(\beta_1, \beta_2, \dots)$ be the matrix given by (3.1) or (3.3) and $c = (c_1, \dots, c_n) \in \mathbb{R}^n$. For every $\rho \in \mathcal{S}_n$, the weighted sum of eigenvalues is defined by

$$\mu_\rho(\theta) = \sum_{k=0}^{n-1} c_{\rho(k+1)} \lambda_k(\theta).$$

Then for $n = 2m$

$$\begin{aligned} \mu_\rho(\theta) &= a_m \left(\sum_{k=0}^{n-1} c_{k+1} (-1)^k \right) \\ &\quad + \sum_{j=1}^{m-1} \Re \left(\exp \left(i \frac{2j\theta}{n} \right) \beta_{m-j} \sum_{k=0}^{n-1} c_{\rho(k+1)} (-1)^k \exp \left(-i \frac{2jk\pi}{n} \right) \right) \end{aligned}$$

and for $n = 2m - 1$

$$\mu_\rho(\theta) = \sum_{j=1}^{m-1} \Re \left(\exp \left(i \frac{(2j-1)\theta}{n} \right) \beta_{m-j} \sum_{k=0}^{n-1} c_{\rho(k+1)} (-1)^k \right. \\ \left. \times \exp \left(-i \frac{(2j-1)k\pi}{n} \right) \right).$$

In the case $n = 2m$ and $1 \leq j \leq m - 1$, we set

$$\beta_{m-j}^{[\rho]} = \beta_{m-j} \sum_{k=0}^{n-1} c_{\rho(k+1)} (-1)^k \exp \left(-i \frac{2jk\pi}{n} \right)$$

and

$$\beta_m^{[\rho]} = a_m^{[\rho]} = a_m \sum_{k=0}^{n-1} (-1)^k c_{\rho(k+1)}.$$

In the case $n = 2m - 1$ and $1 \leq j \leq m - 1$, we set

$$\beta_{m-j}^{[\rho]} = \beta_{m-j} \sum_{k=0}^{n-1} c_{\rho(k+1)} (-1)^k \exp \left(-i \frac{(2j-1)k\pi}{n} \right).$$

Then we have

$$\mu_\rho(\theta) = \lambda \left(A \left(\beta_1^{[\rho]}, \dots, \beta_{[n/2]}^{[\rho]} \right) \right)_{(0)}(\theta), \quad (3.8)$$

the eigenvalue ($k = 0$) of $A(\beta_1^{[\rho]}, \dots, \beta_{[n/2]}^{[\rho]})$ in Theorem 3.1. In the following, we prove an induction for the c -numerical range of A in terms of the convex hull of the classical numerical ranges of matrices of the same type.

Theorem 3.3. *Let $A = A(\beta_1, \dots, \beta_{[n/2]})$ be an $n \times n$ nilpotent Toeplitz matrix given by (3.1) or (3.3) and $c = (c_1, \dots, c_n) \in \mathbb{R}^n$. Then $W_c(A)$ is the convex hull of the $(n-1)!$ numerical ranges $W(A(\beta_1^{[\rho]}, \dots, \beta_{[n/2]}^{[\rho]}))$ where ρ runs over \mathcal{S}_n/T_n and T_n is the subgroup of \mathcal{S}_n consisting of all cyclic permutations.*

Proof. If $(c'_1, c'_2, \dots, c'_n) = (c_n, c_1, \dots, c_{n-1})$, then we have

$$\sum_{k=0}^{n-1} c'_{k+1} \lambda_k(\theta) = \sum_{k=0}^{n-1} c_{k+1} \lambda_{k+1}(\theta) = - \sum_{k=0}^{n-1} c_{k+1} \lambda_k(\theta - \pi),$$

and hence

$$\begin{aligned}
 & \left\{ \left[\left(\sum_{k=0}^{n-1} c'_{k+1} \lambda_k(\theta), -\cos \theta, \sin \theta \right) \right] \in \mathbb{CP}^2 : \theta \in \mathbb{R} \right\} \\
 &= \left\{ \left[\left(-\sum_{k=0}^{n-1} c_{k+1} \lambda_k(\theta - \pi), -\cos \theta, \sin \theta \right) \right] : \theta \in \mathbb{R} \right\} \\
 &= \left\{ \left[\left(\sum_{k=0}^{n-1} c_{k+1} \lambda_k(\theta - \pi), \cos \theta, -\sin \theta \right) \right] : \theta \in \mathbb{R} \right\} \\
 &= \left\{ \left[\left(\sum_{k=0}^{n-1} c_{k+1} \lambda_k(\theta - \pi), -\cos(\theta - \pi), \sin(\theta - \pi) \right) \right] : \theta \in \mathbb{R} \right\} \\
 &= \left\{ \left[\left(\sum_{k=0}^{n-1} c_{k+1} \lambda_k(\theta), -\cos \theta, \sin \theta \right) \right] : \theta \in \mathbb{R} \right\}.
 \end{aligned}$$

It is obvious that $\lambda(\theta) = \sum_j c_j \lambda_j(\theta)$ is the maximum of the set $\{\mu_\rho(\theta) : \rho \in \mathcal{S}_n\}$. By a result in [15], the lines $x = \lambda(\theta)$ and $x = \mu_\rho(\theta)$ are, respectively, the vertical supporting lines of $W_c(A)$ and

$$W(A(\beta_1^{[\rho]}, \dots, \beta_{[n/2]}^{[\rho]})),$$

the assertion then follows from (3.7) and (3.8). \square

The following conclusion is immediate from Theorem 3.3.

Theorem 3.4. *Let $A = A(\beta_1, \dots, \beta_{[n/2]})$ and $c \in \mathbb{R}^n$, and let*

$$B = \oplus \{A(\beta_1^{[\rho]}, \dots, \beta_{[n/2]}^{[\rho]}) : \rho \in \mathcal{S}_n/T_n\}.$$

Then B is an $(n!) \times (n!)$ nilpotent matrix satisfying

$$W(B) = W_c(A(\beta_1, \dots, \beta_{[n/2]})).$$

4. Some examples

Theorem 3.2 shows that the boundary generating curve of the c -numerical range of the matrix (3.1) or (3.3) is a rational curve. In this section we consider several specific matrices, compute their boundary generating curves and discuss their rationality.

First we consider a nilpotent matrix A neither of the form (3.1) nor (3.3) for which the boundary generating curve is rational.

Example 1. Let

$$A = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then we compute that

$$\begin{aligned} F(t, x, y) &= 64 \det(tI_6 + (x/2)(A + A^*) - i(y/2)(A - A^*)) \\ &= 64t^6 - 96t^4x^2 + 16t^3x^3 + 24t^2x^4 - x^6 - 96t^4y^2 + 16t^3xy^2 \\ &\quad + 48t^2x^2y^2 - 3x^4y^2 + 24t^2y^4 - 3x^2y^4 - y^6. \end{aligned}$$

The affine curve $\Gamma = \{(x, y) \in \mathbb{R}^2 : F(1, x, y) = 0\}$ is parametrized by

$$\begin{aligned} x &= \frac{2(1 + \sqrt{2} \sin \theta)(3 - 8 \cos^2 \theta + 2\sqrt{2} \sin \theta)}{3 + 2\sqrt{2} \sin \theta}, \\ y &= \frac{2\sqrt{2} \cos \theta(9 - 8 \cos^2 \theta + 6\sqrt{2} \sin \theta)}{3 + 2\sqrt{2} \sin \theta}, \end{aligned}$$

$\theta \in \mathbb{R}$. The point $(x, y) = (2(1 + \sqrt{2}), 0)$ is a non-singular point of Γ . Thus the polynomial F is irreducible in $\mathbb{C}[t, x, y]$ and Γ is a rational curve. Hence the numerical range $\mathcal{W}(A)$ is the convex hull of a real rational curve.

Next we consider an example of a 4×4 tridiagonal matrix A for which the numerical range $\mathcal{W}(A)$ is the convex hull of two ellipses.

Example 2. Let

$$A = \begin{pmatrix} 0 & 1+i & 0 & 0 \\ 1 & 0 & -i & 0 \\ 0 & 1 & 0 & 1+i \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Then the matrix A is unitarily irreducible and we find that

$$\begin{aligned} F(t, x, y) &= 16 \det(tI_4 + (x/2)(A + A^* - i(y/2)(A - A^*)) \\ &= 16t^4 - 48t^2x^2 + 25x^4 - 16t^2xy + 40x^3y - 16t^2y^2 \\ &\quad + 26x^2y^2 + 8xy^3 + y^4 = F_1(t, x, y)F_2(t, x, y), \end{aligned}$$

where

$$\begin{aligned} F_1(t, x, y) &= (-4t^2 - 2\sqrt{2}tx + 5x^2 + 2\sqrt{2}ty + 4xy + y^2), \\ F_2(t, x, y) &= (-4t^2 + 2\sqrt{2}tx + 5x^2 - 2\sqrt{2}ty + 4xy + y^2). \end{aligned}$$

For each $j = 1, 2$, the dual curve of $F_j(t, x, y) = 0$ is $G_j(t, x, y) = 0$, where

$$\begin{aligned} G_1(t, x, y) &= -t^2 - 6\sqrt{2}tx + 6x^2 + 14\sqrt{2}ty - 12xy + 22y^2, \\ G_2(t, x, y) &= -t^2 + 6\sqrt{2}tx + 6x^2 - 14\sqrt{2}ty - 12xy + 22y^2. \end{aligned}$$

The affine curves

$$\Gamma_j = \{x + iy : (x, y) \in \mathbb{R}^2, G_j(1, x, y) = 0\}$$

satisfy $\Gamma_1 = -\Gamma_2$, and Γ_2 is parametrized by

$$x = \frac{1}{2\sqrt{10}}(-\sqrt{5} + 3\sqrt{6}\cos\theta - \sin\theta),$$

$$y = \frac{1}{2\sqrt{10}}(\sqrt{5} + \sqrt{6}\cos\theta + 3\sin\theta),$$

$0 \leq \theta \leq 2\pi$. The numerical range $W(A)$ is then the convex hull of these two ellipses Γ_1 and Γ_2 .

Remark. Example 2 is a slight modification of the example mentioned in [5, p. 214]. There are other 4×4 unitarily irreducible matrices A constructed in [14, p. 333] with $W(A)$ being the convex hull of two circles.

The following example is a 4×4 real matrix whose numerical range is the convex hull of a rational curve.

Example 3. Consider the matrix

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \sqrt{r^6 - 1} \\ r^3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

where $r > 1$. We compute that

$$\begin{aligned} F(t, x, y) &= 16 \det(tI_4 + (x/2)(A + A^*) - i(y/2)(A - A^*)) \\ &= 16t^4 - 4t^2x^2 - 8r^6t^2x^2 + 4r^3tx^3 - r^6x^4 + r^{12}x^4 - 4t^2y^2 - 6r^6t^2y^2 \\ &\quad - 12r^3txy^2 - 2r^6x^2y^2 + 2r^{12}x^2y^2 - r^6y^4 + r^{12}y^4. \end{aligned}$$

Then the affine curve $\Gamma = \{(x, y) \in \mathbb{R}^2 : F(1, x, y) = 0\}$ is parametrized by

$$x = \frac{-2}{r^6 - 1}(\cos \theta + r^3 \cos(2\theta)),$$

$$y = \frac{-2}{r^6 - 1}(\sin \theta - r^3 \sin(2\theta)),$$

$\theta \in \mathbb{R}$. The curve Γ has a non-singular point $(x, y) = (-2/(r^3 - 1), 0)$. Thus the polynomial F is irreducible and Γ is a rational curve. Hence the numerical range $W(A)$ is the convex hull of a real rational curve. The matrix A has eigenvalues $r, r\omega, r\omega^2, 0$, where $\omega = -1/2 + i\sqrt{3}/2$. Further, the matrix ωA is unitarily equivalent to the matrix A .

It is known that if the graph which corresponds blockwise to an upper triangular nilpotent matrix A is a tree, then the numerical range $W(A)$ is a circular disc. However, the boundary generating curve of the numerical range of a blockwise nilpotent Toeplitz matrix may not be rational. If we replace the entries β_j of the $n \times n$ complex matrix in $A = A(\beta_1, \dots)$ by $m \times m$ mutually commuting normal matrices B_j for which

$$\text{Spec}(B_j) = \{\beta_{j,1}, \beta_{j,2}, \dots, \beta_{j,m}\},$$

then we have a matrix unitarily similar to the direct sum of the matrices

$$A_j = A(\beta_{1,j}, \beta_{2,j}, \dots, \beta_{[n/2],j})$$

for $1 \leq j \leq m$. Thus the boundary generating curve of the numerical range of a blockwise nilpotent Toeplitz matrix with mutually commuting normal blocks is rational. The following example shows that the condition “mutually commuting” is not removable.

Example 4. Let A be the 8×8 real nilpotent matrix defined by

$$A = \begin{pmatrix} 0_2 & B & H & B \\ 0_2 & 0_2 & B & H \\ 0_2 & 0_2 & 0_2 & B \\ 0_2 & 0_2 & 0_2 & 0_2 \end{pmatrix},$$

where $0_2, B, H$ are 2×2 matrices defined by

$$0_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then the boundary generating curve of $W(A)$ is irreducible and has genus 1 and is therefore not a rational curve. The irreducibility and computation of the genus are rather complicated that involve the combinatorial method and Bezout theorem (for details, see [7]).

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